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Quantum Pushdown Automata

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Quantum automata are mathematical models for quantum computing. We analyze the existing quantum pushdown automata, propose a q quantum pushdown automata (qQPDA), and partially clarify their connections. We emphasize some advantages of our qQPDA over others. We demonstrate the equivalence between qQPDA and another QPDA. We indicate that qQPDA are at least as powerful as the QPDA of Moore and Crutchfield with accepting words by empty stack. We introduce the quantum languages accepted by qQPDA and prove that every η-q quantum context-free language is also an η'-q quantum context-free language for any $\eta \in (0, 1)$ and $\eta' \in (0, 1)$.

KEY WORDS: quantum computation; Hilbert spaces; pushdown automata.

1. INTRODUCTION

In the early 1980s, pioneers in quantum computation, such as Benioff (1980), Feynman (1982, 1986), and Deutsch (1985), initially considered the necessity for constructing quantum computers. Benioff (1980) showed that computing devices obeying the principles of quantum physics, that is, unitary quantum evolution, are at least as powerful as a classical computer. By taking a different approach, Feynman (1982, 1986) gave an argument which suggested that the models of computation according to quantum mechanical processes might be beyond any traditional computing machines for solving some problems. Afterwards, Deutsch (1985) proposed the so-called Church–Turing principle and defined quantum computational models-quantum Turing machines. In recent years, quantum computation and quantum information have become a more and more active research field (Nielsen and Chuang, 2000). To a certain extent, this may originate from some discoveries of algorithms Shor (1994) which are ineffective on classical computers. However, there are some limitations and restrictions in quantum computation and information due to the linearity and unitarity of quantum mechanics, for examples,

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the quantum no-cloning theorem (Wootters and Zurek, 1982) and the quantum no-deleting principle (Pati and Braunstein, 2000). So it is of importance to clarify the power of quantum computer (Bernstein and Vazirani, 1997; Yao, 1993).

Quantum automata are some simple mathematical models of computation, so dealing with such models may contribute to further understanding the strengths and weaknesses of quantum computing. Actually, there are considerable literature in this research area (Gudder, 1999, 2000; Kondacs and Watrous, 1997; Moore and Crutchfield, 2000) (The more literature is referred to Gruska, (1999) therein). Because of the unitarity and linearity of quantum physics, there are some essential differences between quantum automata and traditional ones. For examples, Moore and Crutchfield (Moore and Crutchfield, 2000) showed that there are regular languages but not quantum regular languages, and they also showed that there are quantum context-free languages which are not context-free. In Qui (2002) we demonstrated that there is an essential distinction between the sequential quantum machines proposed by Gudder (2000) and the stochastic sequential machines (Paz, 1971). Besides, there are various quantum variants for each classical machine, such as quantum finite-state automata having been differently defined in Gudder (1999), Kondacs and Watrous (1997), and Moore and Crutchfield (2000). The relationship between the Moore and Crutchfield's quantum finite automata and the Kodacs and Watrous' ones is referred to Gruska (1999), while the connections of Gudder's quantum finite automata with those by Moore and Crutchfield are relatively clearer and we discussed them in another paper (in Chinese). In a word, quantum generalizations for classical models of computation are plentiful, and we naturally hope to establish a more suitable framework of quantum automata theory.

As one of the most important quantum computational models after quantum finite automata, quantum pushdown automata (QPDA) have been considerably discussed (Golovkins, 2000; Gudder, 2000; Moore and Crutchfield, 2000). In this paper, we aim to compare partially their connections, and present a QPDA more appropriate to a certain extent. QPDA were first introduced in Moore and Crutchfield (2000), but mainly the so-called generalized QPDA, in which the evolution operators are not required to be unitary or, more relaxedly, isometric. As we know, computation on quantum computers must be reversible, so we are more interested in the unitary evolution automata. By using the definition of quantum finite automata in Kondacs and Watrous (1997), Golovkins (2000) introduced QPDA in an nonequivalent way, compared with those by Moore and Crutchfield (2000), in a way, just as the quantum finite automata defined by Kondacs and Watrous (1997) are different from those by Moore and Crutchfield (2000). Although the QPDA by Golovkins (2000) satisfy the unitary condition on evolution, the transition amplitude function is required to fulfil the so-called well-formedness conditions that are considerably complicated and tedious. So we also hope to simplify them. Notably, Gudder (2000) also presented the definition of QPDA by directly generalizing the classical deterministic pushdown automata (Hopcroft and Ullman, 1979).

However, the transition operators defined in Gudder (2000) are just isometric but not unitary. Therefore in what follows, our purpose is to attempt to establish a more satisfactory QPDA, and to deal with some of its properties.

The organization of this paper is as follows: In Section 2, we propose q quantum pushdown automata (qQPDA) and also preliminarily compare them with the existing QPDA by Moore and Crutchfield (2000), Gudder (2000), and Golovkins (2000). Particularly, we demonstrate the equivalence between qQPDA and QPDA by Moore and Crutchfield (2000) with accepting words by control states. We also indicate that qQPDA are at least as powerful as QPDA defined by Moore and Crutchfield (2000) with accepting words by empty stack. In Section 3, we introduce the quantum languages accepted by qQPDA, that is, η-q quantum context-free language, and prove that every η-q quantum context-free language is also η'-q quantum context-free language for any $\eta \in [0, 1)$ and $\eta' \in (0, 1)$. Finally, in Section 4, we summarize the results obtained and propose some open problems for further study.

2. THE EQUIVALENCE BETWEEN QPDA

For convenience to understand our definition, let us first briefly recall classical pushdown automata (Hopcroft and Ullman, 1979).

A pushdown automata M is a system $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where Q is a finite set of states; Σ is an input alphabet; Γ is a stack' alphabet; $q_0 \in Q$ and $Z_0 \in \Gamma$ are called initial state and initial stack symbol, respectively; $F \subseteq Q$ is the set of final states; and δ is a mapping from $Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma$ to $\mathcal{P}(Q \times \Gamma^*)$. Furthermore, if the mapping δ satisfies (1) $\delta(q, a, Z)$ contains at least one element for any $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$ and $Z \in \Gamma$, and (2) $\delta(q, \epsilon, Z)$ being nonempty implies that $\delta(q, a, Z)$ is nonempty and vice versa, for any $q \in Q$, $Z \in \Gamma$ and $a \in \Sigma$, then M is called deterministic.

There are two fashions to define the language $L(M)$ accepted by M, that is, one is by final state and the other by empty stack. As we know, they are exactly equivalent, but for QPDA the problem is still open. Notably, in Moore and Crutchfield (2000) the authors showed that quantum language accepted by empty stack is also accepted by control state, but the other hand is not yet clear. In this paper, we mainly consider the case of accepting language by control states. Now we give a definition of QPDA. To avoid confusing with others, we call it qQPDA.

Definition 1. A q qQPDA is a six-tuple $\mathcal{A} = \langle Q, \Sigma, \Gamma, \delta, S, Q_f \rangle$ where

(i) Q is a finite set of states;
(ii) Σ is an input alphabet;
(iii) Γ is a stack alphabet;
(iv) $S = \{(p_i, \alpha_i, c_i) : p_i \in Q, \alpha_i \in \Gamma^*, c_i \in \mathbf{C}, i = 1, \ldots, k\}$ for some natural number k with $\Sigma_{i=1}^{k} |c_i|^2 = 1$ is a particular set called the set

of start symbols. (It is different from the classical pushdown automata in which only one initial state $q_0 \in Q$ and one initial stack symbol $Z_0 \in \Gamma$ are required, and so in this definition the initial symbol may be looked as a superposition of some states and strings of stack symbols.)

(v) $Q_f \subseteq Q$ is the set of final states;

(vi) δ is a mapping from $Q \times \Gamma^* \times \Sigma \times Q \times \Gamma^*$ to C satisfying for any $\sigma \in \Sigma$ and any $(p_1, \gamma_1), (p_2, \gamma_2) \in Q \times \Gamma^*$,

(I) $\delta(p_1, \gamma_1, \sigma, p_2, \gamma_2)$ can be nonzero amplitude only if $t\gamma_1 = \gamma_2$, $\gamma_1 = t\gamma_2$, or $\gamma_1 = \gamma_2$ for some $t \in \Gamma$.

(II) $\Sigma_{q\in Q,\gamma\in\Gamma^*}\delta(p_1, \gamma_1, \sigma, q, \gamma)\delta(p_2, \gamma_2, \sigma, q, \gamma)^*$
$$= \begin{cases} 1, & \text{if } (p_1, \gamma_1) = (p_2, \gamma_2), \\ 0, & \text{otherwise.} \end{cases}.$$

(III) $\Sigma_{p\in Q,\gamma\in\Gamma^*}|\delta(p, \gamma, \sigma, p', \gamma')|^2 = 1$ for any $(p', \gamma') \in Q \times \Gamma^*$.

We define a *quantum language* recognized by qQPDA $\mathcal{A}$ as a function

$$f_{\mathcal{A}}(w) = \sum_{q_n\in Q_f,\gamma_n\in\Gamma^*} \Big| \sum_{(p_i,\alpha_i,c_i)\in S} c_i \sum_{q_1,\ldots,q_{n-1}\in Q,\gamma_1,\ldots,\gamma_{n-1}\in\Gamma^*} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \times \delta(q_1, \gamma_1, \sigma_2, q_2, \gamma_2) \cdots \delta(q_{n-1}, \gamma_{n-1}, \sigma_n, q_n, \gamma_n)\Big|^2 \quad (1)$$

for any $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$, and

$$f_{\mathcal{A}}(\epsilon) = \sum_{i\in S_f} |c_i|^2$$

where $S_f = \{i : (p_i, \alpha_i, c_i) \in S, p_i \in Q_f\}$. Particularly, if $S_f = \emptyset$ then $f_{\mathcal{A}}(\epsilon) = 0$.

By repeatedly utilizing (1) and Definition 1 (II), one can verify that for any qQPDA $\mathcal{A}$ with input alphabet Σ, then $f_{\mathcal{A}}(w) \in [0, 1]$ for any words $w \in \Sigma^*$. Now we give an example of qQPDA.

Example 1. Let $\mathcal{A} = \langle\{q_1, q_2\}, \{0, 1\}, \{B, R, G\}, \delta, \{(q_1, R, i)\}, \{q_2\}\rangle$, where δ is defined as follows: For any $\gamma \in \{B, R, G\}^*$

$\delta(q_1, R\gamma, 0, q_1, BR\gamma) = \frac{\sqrt{2}}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $\quad\delta(q_2, R\gamma, 0, q_1, BR\gamma) = \frac{\sqrt{2}}{2}i$

$\delta(q_1, R\gamma, 0, q_1, BR\gamma) = \frac{\sqrt{2}}{2}i$ $\quad\delta(q_2, R\gamma, 0, q_1, GR\gamma) = \frac{\sqrt{2}}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\delta(q_1, B\gamma, 1, q_1, BB\gamma) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ $\quad\delta(q_2, B\gamma, 1, q_1, BB\gamma) = \frac{1}{2} - \frac{i}{2}$

$\delta(q_1, B\gamma, 1, q_1, GB\gamma) = -\frac{1}{2} - \frac{i}{2}$ $\quad\delta(q_2, B\gamma, 1, q_1, GB\gamma) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\delta(q_2, G\gamma, 0, q_2, GG\gamma) = \frac{1}{4} - \frac{\sqrt{7}}{4}i$ $\quad\delta(q_1, G\gamma, 0, q_2, GG\gamma) = \frac{5}{8} + \frac{\sqrt{7}}{8}i$

$\delta(q_2, G\gamma, 0, q_2, \gamma) = -\frac{5}{8} + \frac{\sqrt{7}}{8}i$ $\quad\delta(q_1, G\gamma, 0, q_2, \gamma) = \frac{1}{4} + \frac{\sqrt{7}}{4}i$

One can check readily that the above definition $\mathcal{A}$ is a qQPDA. Furthermore, it is easy to calculate that $f_{\mathcal{A}}(010) = \frac{\sqrt{2}}{2}$, and $f_{\mathcal{A}}(\epsilon) = 1$.

To compare with the QPDA by Moore and Crutchfield (2000) we present a more intuitive definition that is closely related to those defined in Moore and Crutchfield (2000).

Definition 2. A QPDA is a five-tuple $\mathcal{M} = \langle H_Q \otimes H_\Gamma, \sum, |s_{\text{init}}\rangle, U, P(H_{\text{accept}})\rangle$ where H_Q is a finite-dimensional Hilbert space with an orthonormal basis vectors Q (control states) and H_Γ is an infinite-dimensional Hilbert space whose orthonormal basis vectors correspond to finite words over a stack alphabet Γ; Σ is an input alphabet; $|s_{\text{init}}\rangle$ is a unit initial vector of a superposition of finite basis vectors in $H_Q \otimes H_r$, that is, $|s_{\text{init}}\rangle$ has a representation as follows: $|s_{\text{init}}\rangle = \sum_{i=1}^{k} c_i|p_i\rangle \otimes |\alpha_i\rangle$ where $p_i \in Q, \alpha_i \in \Gamma^*, c_i \in \mathrm{C}$ with $\sum_{i=1}^{K} |c_i|^2 = 1$; H_{accept} is a closed subspace of $H_Q \otimes H_\Gamma$ spanned by the set $\{|q\rangle \otimes |\gamma\rangle : q \in Q_{\text{accept}}, \gamma \in \Gamma^*\}$ for some $Q_{\text{accept}} \subseteq Q; U : \Sigma \cup \{\epsilon\} \to \mathcal{U}(H_Q \otimes H_\Gamma)$ satisfies: for any $\sigma \in \Sigma$ and any $(q_1, \gamma_1), (q_2, \gamma_2) \in Q \times \Gamma^*$, the transition amplitude $\langle U(\sigma)(q_1 \otimes \gamma_1), q_2 \otimes \gamma_2\rangle$, that is, $\langle\gamma_2|\langle q_2|U(\sigma)|q_1\rangle|\gamma_1\rangle$ can be nonzero only if $t\gamma_1 = \gamma_2, \gamma_1 = t\gamma_2$, or $\gamma_1 = \gamma_2$ for some $t \in \Gamma$.

We define the language recognized by QPDA $\mathcal{M}$ above as

$$f_{\mathcal{M}}(w) = \|P(H_{\text{accept}})U(w)|s_{\text{init}}\rangle\|^2$$

for any $w \in \Sigma^*$.

Remark 1. In Moore and Crutchfield (2000) H_{accept} in Definition 2 is defined as a closed subspace spanned by $\{|q\rangle \otimes \{\epsilon\} : q \in Q_{\text{accept}}\}$ for some $Q_{\text{accept}} \subseteq Q$, and in this case we call $\mathcal{M}$ an MCQPDA here. Meanwhile, it is well worth noticing that H_{accept} in Definition 2 is exactly equal to the subspace spanned by $\{|q\rangle \otimes |\gamma\rangle : q \in Q_{\text{accept}}, \gamma \in \Gamma^*\}$ since Γ^* is an orthonormal basis of H_Γ.

Lemma 1. (*see Lemma 13 in Moore and Crutchfield (2000)*) *If a quantum language is accepted by a QPDA by empty stack* (*in this case, H_{accept} is a closed subspace spanned by* $\{|q\rangle \otimes \{\epsilon\} : q \in Q\}$), *then it is also accepted by some QPDA by control state* (*that is, H_{accept} is a closed subspace spanned by* $\{|q\rangle \otimes |\gamma : q \in Q_{\text{accept}}, \gamma \in \Gamma^*\}$ *for some* Q_{accept}).

It directly follows from Lemma 1 that a quantum language accepted by an MCQPDA is also accepted by some QPDA, but the converse conclusion is not clear yet. As for the relation between qQPDA and QPDA, we have the following theorem which shows that the classes of languages accepted by them are exactly equivalent. So, qQPDA are at least as powerful as QPDA by Moore and Crutchfield with accepting words by empty stack.

Theorem 2. (*1*) *For any qQPDA* $\mathcal{A} = \langle Q, \Sigma, \Gamma, \delta, S, Q_f \rangle$, *there exists QPDA* $\mathcal{M}$ *such that* $f_{\mathcal{A}}(w) = f_{\mathcal{M}}(w)$ *for any* $w \in \Sigma^*$. (*2*) *For any QPDA* $\mathcal{M} = \langle H_Q \otimes H_\Gamma, \Sigma, s_{\text{init}}, U, P(H_{\text{accept}}) \rangle$, *there exists qQPDA* $\mathcal{A}$ *such that* $f_{\mathcal{A}}(w) = f_{\mathcal{M}}(w)$ *for any* $w \in \Sigma^*$.

Proof: We just present the proof for (1), because in fact, the process of proof for (2) is more simple relatively. Suppose that $S = \{(p_i, \alpha_i, c_i) : p_i \in Q, \alpha_i \in \Gamma^*, c_i \in C, i = 1, \ldots, k\}$ with $\sum_{i=1}^k |c_i|^2 = 1$, then construct $\mathcal{M} = \langle H_Q \otimes H_\Gamma, \Sigma, s_{\text{init}}, U, P(H_{\text{accept}}) \rangle$ where $|s_{\text{init}}\rangle = \sum_{i=1}^k c_i |p_i\rangle \otimes |\alpha_i\rangle$, and H_{accept} that is a closed subspace of $H_Q \otimes H_\Gamma$, is spanned by $\{|q\rangle \otimes |\gamma\rangle : q \in Q_f, \gamma \in \Gamma^*\}$; and for any $\sigma \in \Sigma$, $U(\sigma)$ is defined as follows: for any $(q, \gamma) \in Q \times \Gamma^*$,

$$U(\sigma)(|q\rangle \otimes |\gamma\rangle) = \sum_{q' \in Q, \gamma' \in \Gamma^*} \delta(q, \gamma, \sigma, q', \gamma')(|q'\rangle \otimes |\gamma'\rangle). \tag{2}$$

Now we have to show that $U(\sigma)$ can be extended to be a unitary operator on $H_Q \otimes H_\Gamma$. First, it follows from (2) that for any $q_1, q_2 \in Q$ and any $\gamma_1, \gamma_2 \in \Gamma^*$,

$$\begin{aligned}
&\langle U(\sigma)(|q_1\rangle \otimes |\gamma_1\rangle), U(\sigma)(|q_2\rangle \otimes |\gamma_2\rangle)\rangle \\
&= \Bigg\langle \sum_{q_1' \in Q, \gamma_1' \in \Gamma^*} \delta(q_1, \gamma_1, \sigma, q_1', \gamma_1')(|q_1'\rangle \otimes |\gamma_1'\rangle), \\
&\qquad \sum_{q_2' \in Q, \gamma_2' \in \Gamma^*} \delta(q_2, \gamma_2, \sigma, q_2', \gamma_2')(|q_2'\rangle \otimes |\gamma_2'\rangle) \Bigg\rangle \\
&= \sum_{q \in Q, \gamma \in \Gamma^*} \delta(q_1, \gamma_1, \sigma, q, \gamma) \cdot \delta(q_2, \gamma_2, \sigma, q, \gamma)^* \\
&= \langle |q_1\rangle \otimes |\gamma_1\rangle, |q_2\rangle \otimes |\gamma_2\rangle \rangle = \begin{cases} 1, & \text{if}(q_1, \gamma_1) = (q_2, \gamma_2), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

So for $\sum_{i=1}^\infty c_i A_i$ with $\sum_{i=1}^\infty |c_i|^2 < \infty$, where $A_i \in \{|q\rangle \otimes |\gamma\rangle : q \in Q, \gamma \in \Gamma^*\}$, we may define $U(\sigma)(\sum_{i=1}^\infty c_i A_i) = \sum_{i=1}^\infty c_i U(\sigma) A_i$ and easily show that $\|U(\sigma) |\psi\rangle\| = \||\psi\rangle\|$ for any $|\psi\rangle \in H_Q \otimes H_\Gamma$ and any $\sigma \in \Sigma$. Next let us define operator $U(\sigma)'$ over $H_Q \otimes H_\Gamma$ as

$$U(\sigma)'(|q\rangle \otimes |\gamma\rangle) = \sum_{q' \in Q, \gamma' \in \Gamma^*} \delta(q', \gamma', \sigma, q, \gamma)^* |q'\rangle \otimes |\gamma'\rangle$$

for any $(q, \gamma) \in Q \times \Gamma^*$. Then it easily follows from Definition 1 (III) that $\|U(\sigma)' (|q\rangle \otimes |\gamma\rangle)\| = 1$ for any $(q, \gamma) \in Q \times \Gamma^*$, and thus operator $U(\sigma)'$ can be extended to $H_Q \otimes H_\Gamma$ as extending $U(\sigma)$ above. Now we have

$$U(\sigma)' U(\sigma)(|q\rangle \otimes |\gamma\rangle)$$

$$= U(\sigma)' \left(\sum_{q' \in Q, \gamma' \in \Gamma^*} \delta(q, \gamma, \sigma, q', \gamma')(|q'\rangle \otimes |\gamma'\rangle) \right)$$

$$= \sum_{q' \in Q, \gamma' \in \Gamma^*} \left(\delta(q, \gamma, \sigma, q', \gamma') \left(\sum_{q'' \in Q, \gamma'' \in \Gamma^*} \delta(q'', \gamma'', \sigma, q', \gamma')^* (|q''\rangle \otimes |\gamma''\rangle) \right) \right)$$

$$= \sum_{q'' \in Q, \gamma'' \in \Gamma^*} \left(\sum_{q' \in Q, \gamma' \in \Gamma^*} \delta(q, \gamma, \sigma, q', \gamma') \delta(q'', \gamma'', \sigma, q', \gamma')^* \right) (|q''\rangle \otimes |\gamma''\rangle)$$

$$= |q\rangle \otimes |\gamma\rangle$$

and similarly $U(\sigma)U(\sigma)'(|q\rangle \otimes |\gamma\rangle) = |q\rangle \otimes |\gamma\rangle)$. So $U(\sigma)' = U(\sigma)^{-1}$ and thus $U(\sigma)$ is subjective. It follows from the basic properties of Hilbert spaces that operator $U(\sigma)$ is unitary. The remainder of the proof is to show that $f_{\mathcal{A}}(w) = f_{\mathcal{M}}(w)$ for any $w \in \Sigma^*$. Consider two cases.

Case 1. $w = \epsilon$. Recall $S_f = \{i : (p_i, \alpha_i, c_i) \in S, p_i \in Q_f\}$, then

$$f_{\mathcal{A}}(\epsilon) = \sum_{i \in S_f} |c_i|^2$$

and

$$f_{\mathcal{M}}(\epsilon) = \| P(H_{\text{accept}}) |s_{\text{init}}\rangle \|^2$$

$$= \left\| \sum_{q \in Q_f, \gamma \in \Gamma^*} \left\langle \sum_{i=1}^{k} c_i |p_i\rangle \otimes |\gamma_i\rangle, |q\rangle \otimes |\gamma\rangle \right\rangle |q\rangle \otimes |\gamma\rangle \right\|^2$$

$$= \sum_{p_i \in Q_f} |c_i|^2 = f_{\mathcal{A}}(\epsilon).$$

Case 2. $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$. For any $q \in Q$ and $\gamma \in \Gamma^*$, we have

$$\langle U(w)|s_{\text{init}}\rangle, |q\rangle \otimes |\gamma\rangle\rangle$$

$$= \left\langle U(\sigma_n) \cdots U(\sigma_1) \sum_{i=1}^{k} c_i |p_i\rangle \otimes |\alpha_i\rangle, |q\rangle \otimes |\gamma\rangle \right\rangle$$

$$= \sum_{i=1}^{k} c_i \, \langle U(\sigma_n) \ldots U(\sigma_2)$$

$$\times \left(\sum_{q_1 \in Q, \gamma_1 \in \Gamma^*} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1)(|p_1\rangle \otimes |\gamma_1\rangle) \right), |q\rangle \otimes |\gamma\rangle \Big\rangle$$

$$= \sum_{i=1}^{k} c_i \sum_{q_1 \in Q, \gamma_1 \in \Gamma^*} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \Bigg\langle U(\sigma_n \cdots U(\sigma_3)$$

$$\times \left(\sum_{q_2 \in Q, \gamma_2 \in \Gamma^*} \delta(q_1, \gamma_1, \sigma_2, q_2, \gamma_2)(|q_2\rangle \otimes |\gamma_2\rangle) \right), |q\rangle \otimes |\gamma\rangle \Bigg\rangle$$

$$= \sum_{i=1}^{k} c_i \sum_{q_1 \in Q, \gamma_1 \in \Gamma^*} \sum_{q_2 \in Q, \gamma_2 \in \Gamma^*} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \cdot \delta(q_1, \gamma_1, \sigma_2, q_2, \gamma_2)$$

$$\times \langle U(\sigma_n) \cdots U(\sigma_3)(|q_2\rangle \otimes |\gamma_2\rangle), |q\rangle \otimes |\gamma\rangle\rangle$$

$$= \sum_{i=1}^{k} c_i \sum_{q_i \in Q, \gamma_i \in \Gamma^*, i=1,\ldots,n} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \cdot \prod_{j=1}^{n-1}$$

$$\times \delta(q_j, \gamma_j, \sigma_{j+1}, q_{j+1}, \gamma_{j+1}) \langle |q_n\rangle \otimes |\gamma_n\rangle, |q\rangle \otimes |\gamma\rangle\rangle$$

$$= \sum_{i=1}^{k} c_i \sum_{q_i \in Q, \gamma_i \in \Gamma^*, i=1,\ldots,n-1} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \cdot \prod_{j=1}^{n-2}$$

$$\times \delta(q_j, \gamma_j, \sigma_{j+1}, q_{j+1}, \gamma_{j+1}) \cdot \delta(q_{n-1}, \gamma_{n-1}, \sigma_n, q, \gamma).$$

So

$$\begin{aligned} f_{\mathcal{M}}(w) &= \| P(H_{\text{accept}}) U(w) |s_{\text{init}}\rangle \|^2 \\ &= \sum_{q \in Q, \gamma \in \Gamma^*} |\langle U(w)|s_{\text{init}}\rangle, |q\rangle \otimes |\gamma\rangle\rangle|^2 \\ &= \sum_{q \in Q, \gamma \in \Gamma^*} \Bigg| \sum_{i=1}^{k} c_i \sum_{q_i \in Q, \gamma_i \in \Gamma^*, i=1,\ldots,n-1} \delta(p_i, \alpha_i, \sigma_1, q_1, \gamma_1) \\ &\quad \times \prod_{j=1}^{n-2} \delta(q_j, \gamma_j, \sigma_{j+1}, q_{j+1}, \gamma_{j+1}) \cdot \delta(q_{n-1}, \gamma_{n-1}, \sigma_n, q, \gamma) \Bigg|^2 \\ &= f_{\mathcal{A}}(w). \end{aligned}$$

By combining the above two cases, we complete the proof. □

Remark 2. By utilizing Theorem 2 (1), one can straightforward construct a QPDA $\mathcal{M}$ that is equivalent to the qQPDA defined as Example 1 as follows: $\mathcal{M} = \langle H_Q \otimes H_\Gamma, \Sigma, s_{\text{init}}, U, P(H_{\text{accept}})\rangle$ where H_Q and H_Γ are two closed spaces spanned by $Q = \{|q_1\rangle, |q_2\rangle\}$ and $\{|\gamma\rangle : \gamma \in \{B, R, G\}^*\}$, respectively; H_{accept} is a closed space of $H_Q \otimes H_\Gamma$ and it is spanned by $\{|q_2\rangle \otimes |\gamma\rangle : \gamma \in \Gamma^*\}$; $\Sigma = \{0, 1\}$; $|s_{\text{init}}\rangle = |q_1\rangle$; and U is defined in terms of Eq. (2) and the δ in Example 1.

Remark 3. From the above proof it is seen that the transition operators (please see (2)) derived from qQPDA can be extended to be unitary. In Section 1, we mentioned various QPDA in the literature. To see the more advantages of the qQPDA, we further outline briefly some other QPDA by Gudder (2000) and Golovkins (2000). However, the QPDA defined by Golovkins require considerable conditions, so we refer to Golovkins (2000) for the details. In Gudder (2000), a QPDA is defined as a four-tuple $\mathcal{A} = (Q, \Sigma, \Gamma, \delta)$, where Q, Σ, Γ are as in qQPDA (see Definition 1), and transition amplitude function δ from $Q \times \Gamma \cup \{\epsilon\} \times \Sigma \times Q \times \Gamma \cup \{p\}$ where p means to pop a stack symbol off, to C satisfies

$$\sum_{r,t} \delta(s, v, x, r, t)\delta(s', v, x, r, t)^* = \delta_{s,s'}, \tag{3}$$

for every $v \in \{\epsilon\} \cup \Gamma$;

$$\sum_{r} \delta(s, v, x, r, p)\delta(s', v', x, r, p)^* = 0, \tag{4}$$

for every $v, v' \in \{\epsilon\} \cup \Gamma$, with $v \neq v'$; and

$$\sum_{r} \delta(s, v, x, r, t)\delta(s', v', x, r, p)^* = 0, \tag{5}$$

for every $t \in \Gamma$, $v \in \{\epsilon\} \cup \Gamma$, $v' \in \Gamma$. Gudder showed that the transition operators defined in terms of δ are isometric if and only if the above three conditions (3)–(5) hold. So the transition operators in Gudder's QPDA are not unitary but just isometric, and notably, to a certain extent, Gudder's QPDA may be looked as a special case of qQPDA.

3. QUANTUM LANGUAGES

In this section, we investigate some of the properties of quantum languages recognized by qQPDA.

Definition 3. For any $\eta \in [0, 1)$, an *η-q quantum context-free language* (η- qQCFL) is defined as the set $\{w : f_{\mathcal{A}}(w) > \eta\}$ denoted by $L_\eta(\mathcal{A})$ for some qQPDA $\mathcal{A}$.

Theorem 3. *If $L = L_\eta(\mathcal{A})$ is an η-qQCFL for $0 \leq \eta < 1$, then L is also an η'-qQCFL for every $0 < \eta' < 1$.*

To prove Theorem 3 we need a lemma which shows that under certain conditions, a weighted sum $af + bg$, where $a + b = 1$, of qQCFLs f and g is also a qQCFL.

Lemma 4. *Let $\mathcal{A}_i = \langle Q_i, \Sigma, \Gamma, \delta_i, S_i, Q_f^{(i)}\rangle$ be qQPDA, $i = 1, 2$, where $Q_1 \cap Q_2 = \emptyset$. Then for any $a, b \in C$ with $|a|^2 + |b|^2 = 1$, there exists qQPDA $\mathcal{A}$ with input alphabet Σ such that*

$$f_{\mathcal{A}}(w) = |a|^2 f_{\mathcal{A}_1}(w) + |b|^2 f_{\mathcal{A}_2}(w). \tag{6}$$

Proof: Assume that $S_i = \{(p_j^{(i)}, \alpha_j^{(i)}, c_j^{(i)}) : p_j^{(i)} \in Q_i, \alpha_j^{(i)} \in \Gamma^*, c_j^{(i)} \in C, j = 1, \ldots, k_i\}$ with $\sum_{j=1}^{k_i} |c_j^{(i)}|^2 = 1, i = 1, 2$. Let $\mathcal{A} = \langle Q_1 \cup Q_2, \Sigma, \Gamma, \delta, S, Q_f^{(1)} \cup Q_f^{(2)}\rangle$ where $S = \{(p_i^{(1)}, \alpha_j^{(1)}, ac_j^{(1)}) : p_j^{(1)} \in Q_1, \alpha_j^{(1)} \in \Gamma^*, c_j^{(1)} \in \mathrm{C}, j = 1, \ldots, k_1\} \cup \{(p_j^{(2)}, \alpha_j^{(2)}, ac_j^{(2)}) : p_j^{(2)} \in Q_2, \alpha_j^{(2)} \in \Gamma^*, c_j^{(2)} \in \mathbf{C}, j = 1, \ldots, k_2\}$, δ is defined as follows: for any $(q_i, \gamma_i) \in (Q_1 \cup Q_2) \times \Gamma^*$ and any $\sigma \in \Sigma, i = 1, 2$,

$$\delta(q_1, \gamma_1, \sigma, q_2, \gamma_2) = \begin{cases} \delta_1(q_1, \gamma_1, \sigma, q_2, \gamma_2), & \text{if} \quad q_1, q_2 \in Q_1, \\ \delta_1(q_1, \gamma_1, \sigma, q_2, \gamma_2), & \text{if} \quad q_1, q_2 \in Q_2, \\ 0, & \quad\quad \text{otherwise.} \end{cases}$$

Now S satisfies $\sum_{j=1}^{k} |ac_j^{(1)}|^2 + \sum_{j=1}^{k} |bc_j^{(2)}|^2 = a^2 + b^2 = 1$ and it is easy to check that δ meets the conditions (I), (II), and (III) in Definition 1. So $\mathcal{A}$ is a qQPDA. By utilizing the definitions of $f_{\mathcal{A}}$, $f_{\mathcal{A}_1}$, and $f_{\mathcal{A}_2}$, one has no difficulty in getting (6) and we omit the details here. □

The proof of Theorem 3: Assume that $\mathcal{A} = \langle Q, S, \Sigma, \delta, Q_f\rangle$ where $S = \{(p_i, \alpha_i, c_i) : p_i \in Q, \alpha_i \in \Gamma^*, c_i \in \mathbf{C}, i = 1, \ldots, k\}$ with $\sum_{i=1}^{k} |c_i|^2 = 1$. We discuss it by two cases.

Case 1. $0 < \eta' < \eta < 1$. Let $\mathcal{A}_1 = \langle\{q_0, q_2\}, \Sigma, \Gamma, \delta_1, S_1, Q_f^{(1)}\rangle$ where $\{q_0, q_1\} \cap Q = \emptyset$, $S_1 = \{(q_0, \alpha_0, c_0) : \alpha_0 \in \Gamma^*, c_0 \in C\}$ with $|c_0| = 1$, $Q_f^{(1)} = \{q_1\}$ and δ_1 is defined as follows: for any $\sigma \in \Sigma$, and any $\gamma \in \Gamma^*$, $\delta_1(q_0, \gamma, \sigma, q_0, \gamma) = \delta_1(q_1, \gamma, \sigma, q_1, \gamma) = 1$, and 0 otherwise. Then by a simple calculation one can see that $\mathcal{A}_1$ is a qQPDA and $f_{\mathcal{A}_1}(w) = 0$ for any $w \in \Sigma^*$. According to Lemma 4, for any $a, b \in \mathbf{C}$ with $|a|^2 + |b|^2 = 1$, we may construct a qQPDA $\mathcal{A}' = \langle Q \cup \{q_0, q_1\}, \Sigma, \Gamma, \delta', S', Q_f \cup Q_f^{(1)}\rangle$ where $S' = \{(p_i, \alpha_i, ac_i), (q_0, \alpha_0, bc_0) : p_i, \alpha_i, c_i$ as in $S, i = 1, \ldots, k\}$ and δ' is defined as δ in the proof of Lemma 4. Then it easily follows from the definition of $f_{\mathcal{A}'}$ that for any $w \in \Sigma^*$,

$$f_{\mathcal{A}'}(w) = |a|^2 f_{\mathcal{A}}(w) + |b|^2 f_{\mathcal{A}_1}(w) = |a|^2 f_{\mathcal{A}}(w). \tag{7}$$

Specially, take $a = \sqrt{\frac{\eta'}{\eta}}$, and $b = \sqrt{1 - \frac{\eta'}{\eta}}$, then by (7) $f_{\mathcal{A}}(w) > \eta$ if and only if $f_{\mathcal{A}'}(w) > \eta'$ and therefore $L_\eta(\mathcal{A}) = L_{\eta'}(\mathcal{A}')$.

Case 2. $0 \le \eta < \eta' < 1$. Let $\mathcal{A}_2 = \langle\{q_0\}, \Sigma, \Gamma, \delta_2, S_2, \{q_0\}\rangle$ where $q_0 \in Q$, $S_2 = \{(q_0, \alpha_0, c_0) : \alpha_0 \in \Gamma^*, c_0 \in \mathbf{C}\}$ with $|c_0| = 1$ and δ_2 is defined as follows: for any $\sigma \in \Sigma$ and any $\gamma \in \Gamma^*$, $\delta_2(q_0, \gamma, \sigma, q_0, \gamma) = 1$ and 0 otherwise. Then it is easy to see that $\mathcal{A}_2$ is a qQPDA and $f_{\mathcal{A}_2}(w) = 1$ for any $w \in \Sigma^*$. With $\mathcal{A}$ and $\mathcal{A}_2$ we construct a qQPDA in terms of Lemma 4 as follows: $\mathcal{A}'' = \langle Q \cup \{q_0\}, \Sigma, \Gamma, \delta'', S'', Q_f \cup \{q_0\}\rangle$ where $S'' = \{(p_i, \alpha_i, ac_i), (q_0, \alpha_0, bc_0) : p_i, \alpha_i, c_i$ as in $S, i = 1, \ldots, k\}$ with $|a|^2 + |b|^2 = 1$, δ'' is defined as δ in the proof of Lemma 4. Now since $f_{\mathcal{A}_2}(w) = 1$ for any $w \in \Sigma^*$, we have

$$f_{\mathcal{A}''}(w) = |a|^2 f_{\mathcal{A}}(w) + |b|^2 f_{\mathcal{A}_2}(w) = |a|^2 f_{\mathcal{A}}(w) + |b|^2.$$

Let $|a| = \sqrt{\frac{1-\eta'}{1-\eta}}$ and $|b| = \sqrt{\frac{\eta'-\eta}{1-\eta}}$, then

$$f_{\mathcal{A}''}(w) = \frac{1-\eta'}{1-\eta} f_{\mathcal{A}}(w) + \frac{\eta'-\eta}{1-\eta},$$

and it thus yields that $f_{\mathcal{A}''}(w) > \eta'$ if and only if $f_{\mathcal{A}}(w) > \eta$, that is, $L_\eta(\mathcal{A}) = L_{\eta'}(\mathcal{A}'')$ also holds. Hence, the proof has been completed. □

4. CONCLUSION AND SOME PROBLEMS

We have proposed qQPDA, which, to a certain extent, have some advantages than the other QPDA, which are embodied as follows: (i) quantum languages recognized by MCQPDA (which was defined by Moore and Crutchfield (2000)) with empty stack are also accepted by some qQPDA; (ii) Gudder's QPDA (Gudder, 2000) may be looked as a special case of qQPDA, and particularly the transition operators in qQPDA are exactly unitary, but those on Gudder's ones are just isometric; (iii) compared with QPDA by Golovkins (2000) the conditions for transition function δ on qQPDA are relatively relaxed (indeed, those conditions in Golovkins (2000) are considerably complicated). We also discussed the properties of the languages accepted by qQPDA, and especially prove that every η-q quantum context-free language is also η'-q quantum context-free language for any $\eta \in [0, 1)$ and $\eta' \in (0, 1)$, which is interesting and corresponding to the similar property on quantum regular languages investigated in Gudder (1999). Of course, to a great extent, the strengths or weaknesses of automata may be manifested from their recognizable ability, so there are three problems deserving to be further studied:

1. How to establish appropriate quantum grammars that derive the same class of languages as that by qQPDA? Indeed, we have given quantum regular grammars deriving the same class of languages as that by quantum finite-state automata (Qiu and Ying, manuscript submitted for publication).
2. Languages accepted by quantum finite-state automata in Moore and Crutchfield (2000) are recognized by some MCQPDA (Moore and Crutchfield, 2000) and thus also by some qQPDA. However, there are regular languages but not quantum regular languages, in other words, some regular languages cannot be accepted by any quantum finite-state automaton in Moore and Crutchfield (2000). Naturally, one may ask whether

any regular language can be recognized by qQPDA? Notably, Golovkins (2000) showed that every regular language is recognizable by some QPDA defined by himself.

3. What about the recognizable ability of qQPDA compared with classical pushdown automata (Hopcroft and Ullman, 1979) and probabilistic pushdown automata (Macarie and Ogihara, 1998)?

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